are *generally* covariant. I also contend that these are the only equations that fulfill the following conditions:

1) general covariance.

2) first order compons. T_μ^ν of matter and der[ivations] of the g^{im}'s higher than the second, do not appear in the $\frac{\partial^2 g^{im}}{\partial x_\alpha \partial x_\beta}$'s and the energy tensor.

3) consistent with the "conservation law" of matter without any other restrictions for the $T_{\mu\nu}$'s.

This statement is based above all on the knowledge that aside from the tensors

$$G_{im}$$

$$\text{and } g_{im} \sum_{\alpha\beta} g^{\alpha\beta} G_{\alpha\beta}$$

there are no ⟨*general*⟩ (arbitrary substitutions for covariant) tensors (that satisfy condition (2)). *Your function G vanishes identically*, because—as you can easily calculate—the extension already vanishes, thus all the more so the divergence of the fundamental tensor $g_{\mu\nu}(g^{\mu\nu})$.[4]

Hence it is clear that a consideration according to Hamilton's principle would have to be connected to the V-scalar

$$Q = \sqrt{-g} \sum_{\alpha\beta} g^{\alpha\beta} G_{\alpha\beta},$$

as I already indicated in yesterday's letter. I avoided the somewhat involved computation of the $\frac{\partial Q}{\partial g^{\mu\nu}}$'s and $\frac{\partial Q}{\partial g_\sigma^{\mu\nu}}$'s by setting up the tensor equations directly. But the other way is certainly also workable and even, more elegant mathematically.

With cordial greetings, yours,

A. Einstein.

185. To Paul Ehrenfest

Berlin, Monday, [24 January 1916 or later][1]

Dear Ehrenfest,

Today you should finally be content with me. I am delighted about the great interest you are devoting to this problem. I am not going to support myself at all on the papers but shall calculate everything out for you.[2] Then, if anything should remain incomprehensible, the gap can be easily filled.–

1) Lagrangian form of the equations.

Statement: Let $\sqrt{-g} = 1$. In addition, let $L = g^{\sigma\tau} \left\{ \begin{matrix} \alpha\beta \\ \alpha \end{matrix} \right\} \left\{ \begin{matrix} \tau\alpha \\ \beta \end{matrix} \right\}$. Then follows, if $\mathfrak{L}$ is conceived of as a function of the $g^{\sigma\tau}$'s and $g^{\sigma\tau}_\alpha = \dfrac{\partial g^{\sigma\tau}}{\partial x_\alpha}$:[3]

$$\left.\begin{aligned} \frac{\partial L}{\partial g^{\sigma\tau}} &= \left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} \left\{ \begin{matrix} \tau\alpha \\ \beta \end{matrix} \right\} \\ \frac{\partial L}{\partial g^{\sigma\tau}_\alpha} &= -\left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} \end{aligned}\right\} \qquad (1)$$

Comment: I always omit the summation symbol. An index must always be summed when it appears twice.[4]

Proof: From differentiating $\mathfrak{L}$—always considered as a funct. of $g^{\sigma\tau}$ & $g^{\sigma\tau}_\alpha$—follows

$$dL = \left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} \left\{ \begin{matrix} \tau\alpha \\ \beta \end{matrix} \right\} dg^{\sigma\tau} + 2g^{\sigma\tau} \left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} d \left\{ \begin{matrix} \tau\alpha \\ \beta \end{matrix} \right\}.$$

(Two terms differing only in index name are summarized).

From this follows furthermore from $g^{\sigma\tau} d \left\{ \begin{matrix} \tau\alpha \\ \beta \end{matrix} \right\} = d\left(g^{\sigma\tau} \left\{ \begin{matrix} \tau\alpha \\ \beta \end{matrix} \right\}\right) - \left\{ \begin{matrix} \tau\alpha \\ \beta \end{matrix} \right\} dg^{\sigma\tau}$

$$\begin{aligned} dL &= -dg^{\sigma\tau} \cdot \left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} \left\{ \begin{matrix} \tau\alpha \\ \beta \end{matrix} \right\} + 2 \left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} d\left(g^{\sigma\tau} \left\{ \begin{matrix} \tau\alpha \\ \beta \end{matrix} \right\}\right) \\ &= -dg^{\sigma\tau} \{\ \}\{\ \} + 2 \left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} d\left(g^{\sigma\tau} g^{\beta\lambda} \left[\begin{matrix} \tau\alpha \\ \lambda \end{matrix} \right]\right) \end{aligned}$$

In addition

$$\left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} = \left\{ \begin{matrix} \beta\sigma \\ \alpha \end{matrix} \right\} \cdot \cdot \cdot \qquad (\alpha)$$

Taking into consideration that the second term does not change when the sum of the indices α & β are exchanged simultaneously with λ and τ, the second term is then also equal to

$$\left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} d\left(g^{\sigma\tau} g^{\beta\lambda} \left(\left[\begin{matrix} \tau\alpha \\ \lambda \end{matrix} \right] + \left[\begin{matrix} \lambda\alpha \\ \tau \end{matrix} \right] \right)\right)$$

or equal to $$\left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} d \left(g^{\sigma\tau} g^{\beta\lambda} \frac{\partial g_{\lambda\tau}}{\partial x_\alpha} \right)$$

or equal to[5] $$-\left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} d g_\alpha^{\sigma\tau}$$

For from $g_{\rho\sigma} g^{\sigma\tau} = \delta_\rho^\tau = 1$ or 0 follows

$$\frac{\partial g_{\rho\sigma}}{\partial x_\alpha} g^{\sigma\tau} = -g_{\rho\sigma} \frac{\partial g^{\sigma\tau}}{\partial x_\alpha}, \qquad (\beta)$$

& from this through multiplication by $g^{\rho\lambda}$

$$g^{\rho\lambda} g^{\sigma\tau} \frac{\partial g_{\rho\sigma}}{\partial x_\alpha} = -\frac{\partial g^{\lambda\tau}}{\partial x_\alpha}$$

analogously also

$$g_{\rho\lambda} g_{\sigma\tau} \frac{\partial g^{\sigma\tau}}{\partial x_\alpha} = -\frac{\partial g_{\lambda\tau}}{\partial x_\alpha} \qquad (\beta')$$

It thus follows[6]

$$dL = \left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} \left\{ \begin{matrix} \tau\alpha \\ \beta \end{matrix} \right\} dg^{\sigma\tau} - \left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\} dg_\alpha^{\sigma\tau},$$

from which statement (1) follows. From this it follows that the gravitation equations can be written in the form[7]

$$\frac{\partial}{\partial x_\alpha} \left(\frac{\partial L}{\partial g_\alpha^{\sigma\tau}} \right) - \frac{\partial L}{\partial g^{\sigma\tau}} = -\kappa (T_{\sigma\tau} - \frac{1}{2} g_{\sigma\tau} T) \ldots \qquad (2)$$

2) Conservation laws.

If you multiply (2) by $g_\beta^{\sigma\tau}$, then you obtain upon partial differentiation reformulation of the first term,[8] because of interchangeability of α and β,

$$\frac{\partial}{\partial x_\alpha} \left(g_\beta^{\sigma\tau} \frac{\partial \mathfrak{L}}{\partial g_\alpha^{\sigma\tau}} \right) - \underbrace{\left(\frac{\partial L}{\partial g^{\sigma\tau}} g_\beta^{\sigma\tau} + \frac{\partial L}{\partial g_\alpha^{\sigma\tau}} \frac{\partial g_\alpha^{\sigma\tau}}{\partial x_\beta} \right)}_{\frac{\partial L}{\partial x_\beta}} = -\kappa T_{\sigma\tau} g_\beta^{\sigma\tau}.$$

The second term on the right-hand side disappears because

$$g^{\sigma\tau} \frac{\partial g^{\sigma\tau}}{\partial x_\beta} = -g^{\sigma\tau} \frac{\partial g_{\sigma\tau}}{\partial x_\beta} = -\frac{\partial \lg g}{\partial x_\beta} = 0.$$

Now I write the conservation law of matter by introducing it formally without assuming its validity[9] [[A_μ's are] unknown space functions]

$$\frac{\partial T_\mu^\sigma}{\partial x_\sigma} + \frac{1}{2} g_\mu^{\alpha\tau} T_{\alpha\tau} = A_\mu \ldots \tag{3}$$

The second term on the left-hand side can be converted because of (β) into the form $-\frac{1}{2}\frac{\partial g_{\alpha\tau}}{\partial x_\mu} T^{\alpha\tau}$. The { } can likewise be introduced, which I do not need here, however. With the aid of this, the right-hand side of the last equation can be substituted by

$$2\kappa \frac{\partial T_\beta^\alpha}{\partial x_\alpha} - 2\kappa A_\beta,$$

and you obtain:

$$\frac{\partial}{\partial x_\alpha}\Bigg(\underbrace{\frac{1}{2\kappa}\left[L\delta_\beta^\alpha - g_\beta^{\sigma\tau}\frac{\partial L}{\partial g_\alpha^{\sigma\tau}}\right]}_{t_\beta^\alpha} + T_\beta^\alpha\Bigg) = A_\beta \ldots \tag{4}$$

If the A_β's were to vanish, this would be the conservation equation both for matter and for gravitation. For the moment we shall work with (4). Taking into account (1)[10] it follows from the definition of the t_β^α's

$$t_\alpha^\alpha = t = \frac{1}{\kappa} L \ldots \tag{5}$$

3) *Mixed form of the gravitation equations.* We write the latter in the form

$$-\frac{\partial \left\{ \begin{matrix} \sigma\tau \\ \alpha \end{matrix} \right\}}{\partial x_\alpha} + \left\{ \begin{matrix} \sigma\beta \\ \alpha \end{matrix} \right\}\left\{ \begin{matrix} \tau\alpha \\ \beta \end{matrix} \right\} = -\kappa(T_{\sigma\tau} - \frac{1}{2} g_{\sigma\tau} T) \cdots \tag{2a}$$

and multiply by $g^{\tau\nu}$. The first term on the left yields through partial diff. reformulation

$$-\frac{\partial}{\partial x_\alpha}\left(g^{\tau\nu}\left\{ \begin{matrix} \sigma\tau \\ \alpha \end{matrix} \right\}\right) + \left\{ \begin{matrix} \sigma\tau \\ \alpha \end{matrix} \right\}\frac{\partial g^{\tau\nu}}{\partial x_\alpha},$$

of which the second term can be modified with the aid of the formula (γ), deduced here to the side,[11] so that you obtain

$$-\frac{\partial}{\partial x_\alpha}\left(g^{\tau\nu}\left\{ \begin{matrix} \sigma\tau \\ \alpha \end{matrix} \right\}\right) - g^{\tau\varepsilon}\left\{ \begin{matrix} \varepsilon\alpha \\ \nu \end{matrix} \right\}\left\{ \begin{matrix} \tau\sigma \\ \alpha \end{matrix} \right\} - g^{\nu\varepsilon}\left\{ \begin{matrix} \varepsilon\alpha \\ \tau \end{matrix} \right\}\left\{ \begin{matrix} \sigma\tau \\ \alpha \end{matrix} \right\}.$$

Calculation aid [*Hilfsrechnung*]:

$$\frac{\partial g^{\tau\nu}}{\partial x_\alpha} = -g^{\tau\varepsilon} g^{\nu\xi} \frac{\partial g_{\varepsilon\xi}}{\partial x_\alpha} \text{ (because of } \beta')$$

$$= -g^{\tau\varepsilon} g^{\nu\xi} \left(\begin{bmatrix} \varepsilon\alpha \\ \xi \end{bmatrix} + \begin{bmatrix} \xi\alpha \\ \varepsilon \end{bmatrix} \right)$$

thus

$$\frac{\partial g^{\tau\nu}}{\partial x_\alpha} = -g^{\tau\varepsilon} \left\{ \begin{matrix} \varepsilon\alpha \\ \nu \end{matrix} \right\} - g^{\tau\varepsilon} \left\{ \begin{matrix} \varepsilon\alpha \\ \tau \end{matrix} \right\} \dots \qquad (\gamma)$$

The third of these terms[12] cancels out with the one formed from the second of (2a). Hence you obtain initially

$$\frac{\partial}{\partial x_\alpha} \left(g^{\tau\nu} \left\{ \begin{matrix} \sigma\tau \\ \alpha \end{matrix} \right\} \right) + g^{\varepsilon\tau} \left\{ \begin{matrix} \varepsilon\alpha \\ \nu \end{matrix} \right\} \left\{ \begin{matrix} \tau\sigma \\ \alpha \end{matrix} \right\} = \kappa (T^\nu_\sigma - \frac{1}{2} \delta^\nu_\sigma T) \dots \qquad (6)$$

From the definition of t^α_β and the equations (1) & (5) you obtain

$$t^\alpha_\beta = \frac{1}{2} t \delta^\alpha_\beta + \frac{1}{2\kappa} \left\{ \begin{matrix} \sigma\tau \\ \alpha \end{matrix} \right\} \frac{\partial g^{\sigma\tau}}{\partial x_\beta}.$$

On transforming the second term according to (γ) and combining both the thus formed terms

$$t^\alpha_\beta - \frac{1}{2} \delta^\alpha_\beta t = -\frac{1}{\kappa} g^{\sigma\varepsilon} \left\{ \begin{matrix} \sigma\tau \\ \alpha \end{matrix} \right\} \left\{ \begin{matrix} \varepsilon\beta \\ \tau \end{matrix} \right\} \dots \qquad (7)$$

Disregarding the factor $-\frac{1}{\kappa}$ and the index designation, the right-hand side of (7) corresponds to the second term in (6), so that you can write

$$\frac{\partial}{\partial x_\alpha} \left(g^{\tau\nu} \left\{ \begin{matrix} \sigma\tau \\ \alpha \end{matrix} \right\} \right) = \kappa \left((T^\nu_\sigma + t^\nu_\sigma) - \frac{1}{2} \delta^\nu_\sigma (T + t) \right) \dots \qquad (8)$$

This equation is interesting because it shows that the source of the gravitation lines is determined solely by the sum $T^\nu_\sigma + t^\nu_\sigma$, as must obviously be expected.[13]

(Second sheet)

4) *Proof that the* A_μ*'s vanish.* Now comes the main issue.

a) If (8) is multiplied by δ^σ_ν, then your obtain the scalar equation

$$\frac{\partial}{\partial x_\alpha} \left(g^{\tau\nu} \left\{ \begin{matrix} \nu\tau \\ \alpha \end{matrix} \right\} \right) = -\kappa (T + t) \dots \qquad (9)$$

The left-hand side reads fully as

$$\frac{1}{2}\frac{\partial}{\partial x_\alpha}\left(g^{\nu\tau}g^{\alpha\beta}\left(\frac{\partial g_{\beta\nu}}{\partial x_\tau}+\frac{\partial g_{\beta\tau}}{\partial x_\nu}-\frac{\partial g_{\nu\tau}}{\partial x_\beta}\right)\right).$$

The third term yields nothing, because $g^{\nu\tau}\dfrac{\partial g_{\nu\tau}}{\partial x_\beta}=\dfrac{\partial \lg g}{\partial x_\beta}=0$. The first two coincide through an exchange of ν and τ, such that they can be combined. By applying (β') you obtain finally

$$-\frac{\partial^2 g^{\alpha\tau}}{\partial x_\alpha \partial x_\tau}.$$

From (9) thus results

$$\frac{\partial^2 g^{\alpha\beta}}{\partial x_\alpha \partial x_\beta}-\kappa(T+t)=0\cdots \tag{9a}$$

We execute the operation $\dfrac{\partial}{\partial x_\nu}$ on (8) and allowing for (4) obtain[14]

$$\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\nu}\left(g^{\tau\nu}\left\{\begin{matrix}\sigma\tau\\ \alpha\end{matrix}\right\}\right)=-\frac{1}{2}\frac{\partial(T+t)}{\partial x_\sigma}+\kappa A_\sigma\ldots \tag{10}$$

The left-hand side is more completely

$$\frac{1}{2}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\nu}\left(g^{\tau\nu}g^{\alpha\beta}\left(\frac{\partial g_{\sigma\beta}}{\partial x_\tau}+\frac{\partial g_{\tau\beta}}{\partial x_\sigma}-\frac{\partial g_{\sigma\tau}}{\partial x_\beta}\right)\right).$$

If you exchange in the first term α and ν as well as β and τ, then disregarding the index, it coincides with the third. Only the second remains, which allowing for (β') changes into

$$-\frac{1}{2}\frac{\partial^3 g^{\alpha\nu}}{\partial x_\alpha \partial x_\nu \partial x_\sigma}.$$

That is why you obtain instead of (10)[15]

$$\frac{\partial}{\partial x_\sigma}\left(\frac{\partial^2 g^{\alpha\nu}}{\partial x_\alpha \partial x_\nu}-(T+t)\right)=-2\kappa A_\sigma\ldots \tag{10a}$$

From (9a) and (10a) $A_\sigma=0$ results, that is, according to (3) of the conservation law of matter, as a consequence of the field equations (2).–

You will certainly not encounter any more problems now. Show this thing to Lorentz as well, who also does not yet perceive the need for the structure on the right-hand side of the field equations. I would appreciate it if you would then give these pages back to me, because nowhere else do I have these things so nicely in one place.

With best regards, yours,

Einstein.

186. To Arnold Sommerfeld

[Berlin,] 2 February 1916

Dear Sommerfeld,

I've been cracking my brains about your letter, especially since I must acknowledge quite much of what you say as correct. Fr.[1] is of the "greyhound" breed, more or less, as defined by a good acquaintance of mine. His way of bolting is also not particularly distinguished. I have known this person's weaknesses for a long time[2]—and have also been more or less irritated by him. It is undoubtedly justified to raise the question: Is Einstein right when he attempts to remove all obstacles in this person's career?[3] You answer in the negative. I have thought about the matter thoroughly and also have discussed it with an intelligent and well-intentioned person to whom I had presented the "evidence" and whose objectivity in the matter is entirely beyond doubt.[4]

First to the personal characteristics. I would *not* choose Fr. as an intimate friend of mine but would always maintain a healthy distance from him either way. And yet I come to the conclusion: If the devil were to unseat all of those among our professorial colleagues whose self-criticism and decency are not above that of Freundlich's, then the trusty ranks would be considerably thinned. I would even—*horribile dictu*—fear for your informant S![5] On the other hand, Freundl. offers something worth its weight in gold—an enthusiastic dedication to the problem; that is a rare trait he does not share with very many.

Now to the professional qualities. Freundl. is not really creatively talented, but he is intelligent and resourceful. The greyhound nature of his, mentioned above, comes to a large part from his pounding heart when he investigates a scientifically important issue. We must not forget that Fr. had devised the statistical method that makes possible using fixed stars in addressing the line-shift question. Although the nasty calculation error did slip by him and some other things there are greyhound-like as well (density definition), the overall value of the matter ought not to be forgotten because of it.[6] Errors can be corrected and in time are always corrected. The task involves discovering a way and smoothing it until it becomes passable.